On Generalized Non-Symmetric Recurrent Spaces

by

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Abstract.

In this paper we introduced a Finsler space $F_n$whose Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are satisfied the generalized of recurrence condition with respect to Cartan’s connection parameter Γ^{*i}_{kh}which given by the following conditions $R^i_{jkh}|_l = \lambda_l R^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$and $K^i_{jkh}|_l = \lambda_l K^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, respectively, where $|_l$ is v-covariant differentiation, $\lambda_l$and μ_l are the recurrence vectors field and such spaces are called as *a generalized R^v-recurrent space*and *a generalized K^v-recurrent space*, respectively, denoted them briefly by$G\,R^v$-$R\,F_n$ and $G\,K^v$-$R\,F_n$, respectively.

The purpose of this paper is to devolpe the above spaces by study them in h-isotropic and non-symmetric spaces. We have obtained the v-covariant derivative for Cartan's third andfourth curvature tensorsR^i_{jkh}andK^i_{jkh} , the h(v)-torsion tensor H^i_{kh} and other different tensors, we proved that, R-Ricci tensorR_{jk}andK-Ricci tensorK_{jk} are non-vanishing in our spaces. We obtain different theorems for some tensors satisfied the generalized recurrence conditions of the above spaces.Some conditions have been pointed out which reduce these spaces$(n > 2)$into a Finsler space of curvature tensor. We obtained different theorems for some tensors satisfying separately the conditions of generalized recurrent spaces and established the decomposition of curvature tensors field $\overset{+}{R}{}^i_{jkh}(x,y)$ and $\overset{+}{K}{}^i_{jkh}(x,y)$ in a Finsler space $F_n$equipped with non-symmetric connection of Cartan's third curvature tensorR^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} .

Key words: Generalized R^v-recurrent space, Generalized K^v-recurrent space, Generalized $\overset{+}{R}{}^v$- non symmetric recurrent space, Generalized $\overset{+}{K}{}^v$- non symmetric recurrent space.

Introduction. On account of the different connections of Finsler space, the concept of the recurrent for different curvature tensors have been discussed by M. Matsumoto [7], P.N. Pandey ([10], [11]), R.S.D. Dubey and A.K. Srivastava [5], P.N. Pandey and R.B.Misra [12], P.N. Pandey and V.J. Dwivedi [13], Z. Ahsanand M.Ali [1], R. Verma [20], S. Dikshit [4], F.Y.A. Qasem [16], P.N. Pandey and S. Pal [14]. The generalized recurrent space studied by U.C. De and N. Guha [3], Y.B. Maralebhavi and M. Rathnamma [6]. M. L. Zlatanović and S. M. Minčić [22] whom obtainedidentities for curvature tensors in generalized Finsler space. C. K. Mishra and G. Lodhi [8] discussed C^h-recurrent and C^v-recurrent Finsler spaces of second order and obtained different theorems regarding these spaces, the decomposability of the curvature tensor in recurrent conformal Finsler spaces also, they studied the decomposition of curvature tensor field $\overset{+}{R}{}^i_{jkh}(x,y)$in a Finsler space equipped with non-symmetric connection were study by P. Mishra, K. Srivistava and S. B. Mishra [9].P.N. Pandey, S. Saxena and A.Goswani [15], F.Y.A. Qasem and A.M.A. Al-Qashbari ([17], [18]) and others.

Let us consider an n-dimensional Finsler space $F_n$equipped with the metric function F satisfying the requisite conditions [19]. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameter $\Gamma^{*i}_{jk}$and Berwald's connection parameter G^i_{jk} *. These are symmetric in their lower indices and positively homogeneous of degree zero in the directional argument. The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by

(1.1) $$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases}$$

The vectors y_i and y^isatisfies the following relations

*The indices $i\,, j\,, k\,, \ldots$ assume positive integral values from 1 to n .

(1.2) a) $y_i = g_{ij}\, y^j$ and b) $\dot{\partial}_j y_i = g_{ij}$.

The tensor C_{ijk} *2 defined by

(1.3) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$

is known as *(h) hv - torsion tensor* [7] . It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor $C_{ijk}$are related by

(1.4) a) $C^i_{jk}\, y^j = 0 = C^i_{kj}\, y^j$, b) $y_i\, C^i_{jk} = 0$ and c) $C_{ijk} := g_{hj} C^h_{ik}$.

The (v) hv-torsion tensor C^h_{ik} is also positively homogeneous of degree -1 in the directional argument and symmetric in its lower indices.

É. Cartan deduced the v-covariant derivativefor an arbitrary vector filed X^i with respect to x^k [2]

(1.5) $DX^i = F\, X^i|_k\, D\, l^k + X^i_{|k}\, dx^k + y^k(\dot{\partial}_k X^i)\frac{dF}{F}$,

where

(1.6) $X^i|_k := \dot{\partial}_k X^i + X^r C^i_{rk}$.

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to the above process, i.e.

(1.7) a) $y^i|_k = \delta^i_k$ and b) $g_{ij}|_k = 0$.

The quantitiesH^i_{jkh} and H^i_{kh}form the components of tensors and they called *h-curvature tensorof Berwald* (*Berwald curvature tensor*) and *h(v) –torsion tensor*, respectively and defined as follow:

(1.8) a) $H^i_{jkh} := \partial_j\, G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj}\, G^r_k - h/k\, *$

and

b) $H^i_{kh} := \partial_h\, G^i_k + G^r_k\, C^i_{rh} - h/k$.

*2 Unless stated otherwise. Henceforth all geometric objects are assumed to be functions of line-elements.

They are skew-symmetric in their lower indices, i.e. k and h . They are positively homogeneous of degree zero and one, respectively in their directional argument.

These tensors were constructed initially by mean of the tensor H_h^i , called the *deviation tensor*, given by

(1.9) a) $H_h^i := 2\,\partial_h G^i - \partial_r G_h^i\, y^r + 2\, G_{hs}^i\, G^s - G_s^i\, G_h^s$,

where

b) $\dot{\partial}_k G_h^i = G_{kh}^i$.

The deviation tensor H_h^i is positively homogeneous of degree two in the directional argument.

The quantities H_{jkh}^i , H_{kh}^i and H_k^i are satisfied the following [19]:

(1.10) a) $H_{jkh}^i = \partial_j H_{kh}^i$, b) $H_{kr}^r := H_k$ and c) $H_{jk.h} := g_{jr} H_{hk}^r$.

Cartan's third and fourth curvature tensors are defined as

(1.11) a)

$R_{jkh}^i = \dot{\partial}_h\, \Gamma_{jk}^{*i} + \left(\dot{\partial}_l\, \Gamma_{jk}^{*i}\right) G_h^l + C_{jm}^i \left(\dot{\partial}_k\, G_h^m - G_{kl}^m G_h^l\right) + \Gamma_{mk}^{*i}\, \Gamma_{jh}^{*m} - k/h$

and

b) $K_{rhk}^i := \partial_k \Gamma_{hr}^{*i} + \left(\dot{\partial}_l\, \Gamma_{rk}^{*i}\right) G_h^l + \Gamma_{mk}^{*i} \Gamma_{hr}^{*m} - k/h *$

respectively.

Cartan's third curvature tensorR_{jkh}^i , Cartan's fourth curvature tensorK_{jkh}^i andtheir associate curvature tensors R_{ijkh} and K_{ijkh} , respectively are given by

(1.12) a) $R_{jkh}^i y^j = H_{kh}^i = K_{jkh}^i y^j$, b) $g_{rj}\, R_{ihk}^r = R_{ijkh}$,

c) $R_{jki}^i := R_{jk}$, d) $g_{rj}\, K_{ihk}^r = K_{ijkh}$ and e) $K_{jki}^i := K_{jk}$.

Ricci tensorsR_{jk} and K_{jk} of the curvature tensor $R_{jkh}^i$and K_{jkh}^i , respectively, the curvature vectors R_k and K_k are connected by

(1.13) a) $R_{jk} y^k = R_j$ and b) $K_{jk} y^k = K_k$.

F.Y.A. Qasem and A.M.A. AL-Qashbari [17] discussed a Generalized R^h- Recurrent spacewhose Cartan's third curvature tensor R^i_{jkh} satisfies the generalized recurrence property in the sense of Cartan by using the h-covariant differentiation.

2. On Study of Generalized R^v- Recurrent Space and K^v- Recurrent Space

We shall study some properties of a generalized R^v- recurrent spaceand a generalized K^v- recurrent spacewhose Cartan's third and fourth curvature tensorsR^i_{jkh} andK^i_{jkh}satisfythe following conditions

(2.1) $R^i_{jkh}|_l = \lambda_l R^i_{jkh} + \mu_l (\delta^i_h g_{jk} - \delta^i_k g_{jh})$, $R^i_{jkh} \neq 0$

and

(2.2) $K^i_{jkh}|_l = \lambda_l K^i_{jkh} + \mu_l (\delta^i_h g_{jk} - \delta^i_k g_{jh})$, $K^i_{jkh} \neq 0$,

respectively.

where $|_l$ is v-covariant differentiation (Cartan's first kind covariant differential operator),

λ_l and μ_l are called *recurrence vectors*.

Definition 2.1. A Finsler space F_n whoseCartan's third curvature tensorR^i_{jkh} is satisfying the condition (2.1), where λ_l and μ_l are non-null covariant vectors field, is called *a generalized* R^v-*recurrent space* and the tensor will be called generalized *v -recurrent*, respectively. We shall denote them briefly by$G\ R^v$- $R\ F_n$ and $G\ v$-R , respectively.

Definition 2.2. A Finsler space F_n whoseCartan's fourth curvature tensorK^i_{jkh} is satisfying the condition (2.2), where λ_l and μ_l are non-null covariant vectors field, is called *a generalized* K^v-*recurrent space* and the tensor will be called generalized *v*-

recurrent, respectively. We shall denote them briefly by $G\ K^v$- $R\ F_n$ and $G\ v$-R , respectively.

Transvecting the conditions (2.1) and (2.2) by y^j , using (1.7a), (1.12a), (1.2a) and in view of (1.1), we get

(2.3) $$H^i_{kh}|_l = \lambda_l\, H^i_{kh} + R^i_{lkh} + \mu_l(\, \delta^i_h y_k - \delta^i_k y_h\,)$$

and

(2.4) $$H^i_{kh}|_l = \lambda_l\, H^i_{kh} + K^i_{lkh} + \mu_l(\, \delta^i_h y_k - \delta^i_k y_h\,) \quad ,$$

respectively.

Thus, we conclude

Theorem 2.1. *In$G\ R^v$-$R\ F_n$ and $G\ K^v$-$R\ F_n$,the v-covariant derivative of the first order for the h(v)-torsion tensor H^i_{kh} given by the conditions (2.3) and (2.4), respectively.*

The condition (2.3) and (2.4) can be written as

(2.5) $$R^i_{lkh} = H^i_{kh}|_l - \lambda_l\, H^i_{kh} - \mu_l(\, \delta^i_h y_k - \delta^i_k y_h\,)$$

and

(2.6) $$K^i_{lkh} = H^i_{kh}|_l - \lambda_l\, H^i_{kh} - \mu_l(\, \delta^i_h y_k - \delta^i_k y_h\,) \quad ,$$

respectively.

Thus, we conclude

Theorem 2.2. *In$G\ R^v$-$R\ F_n$ and $G\ K^v$-$R\ F_n$, Cartan's third and fourth curvature tensors R^i_{jkh} and K^i_{jkh} , respectively, defined by any one of the conditions (2.5) or (2.6).*

Theorem 2.3. *Cartan's third curvature tensorR^i_{jkh} in$G\ R^v$- $R\ F_n$ coincides withCartan's fourth curvature tensorK^i_{jkh} in $G\ K^v$- $R\ F_n$ and they are in terms of the h(v)-torsion tensor H^i_{kh} .*

Transvecting the conditions (2.1), (2.2), (2.3) and (2.4) by g_{ip} , using (1.7b), (1.12b), (1.12d), (1.10c) and in view of (1.1), we get

(2.7) $$R_{jpkh}|_l = \lambda_l R_{jpkh} + \mu_l(\, g_{hp} g_{jk} - g_{kp} g_{jh}) \quad ,$$

(2.8) $$K_{jpkh}|_l = \lambda_l K_{jpkh} + \mu_l(\, g_{hp} g_{jk} - g_{kp} g_{jh}) \quad ,$$

(2.9) $$H_{kp.h}|_l = \lambda_l H_{kp.h} + R_{lpkh} + \mu_l(\, g_{hp} y_k - g_{kp} y_h)$$

and

(2.10) $$H_{kp.h}|_l = \lambda_l H_{kp.h} + K_{lpkh} + \mu_l(\, g_{hp} y_k - g_{kp} y_h) \quad ,$$

respectively. Conversely, the transvection of the conditions (2.7), (2.8), (2.9) and (2.10) by g^{ip} , gives us the conditions (2.1), (2.2), (2.3) and (2.4), respectively. Thus, the conditions (2.1), (2.2), (2.3) and (2.4) are equivalent to the conditions (2.7), (2.8), (2.9) and (2.10), respectively. Therefore $G\,R^v$- $R\,F_n$ characterized by the conditions (2.1) or (2.7) and $G\,K^v$- $R\,F_n$characterized by the conditions (2.2) or (2.8), respectively.

Thus, we conclude

Theorem 2.4.*An$G\,R^v$- $R\,F_n$,may the characterized by the condition (2.7).*

Theorem 2.5. *An$G\,R^v$- $R\,F_n$,may the characterized by the condition (2.8).*

Theorem 2.6. *In$G\,R^v$- $R\,F_n$ and $G\,K^v$- $R\,F_n$, thev-covariant derivative of the first order for the associate torsion tensor $H_{kp.h}$ is given by the conditions (2.9) and (2.10), respectively.*

The conditions (2.9) and (2.10) can be written as

(2.11) $$R_{lpkh} = H_{kp.h}|_l - \lambda_l H_{kp.h} - \mu_l\big(g_{hp}y_k - g_{kp}y_h\big)$$

and

(2.12) $$K_{lpkh} = H_{kp.h}|_l - \lambda_l H_{kp.h} - \mu_l\big(g_{hp}y_k - g_{kp}y_h\big) \quad ,$$

respectively.

Thus, we conclude

Theorem 2.7. *In$G\,R^v$- $R\,F_n$ and $G\,K^v$- $R\,F_n$, the associate curvature tensors $R_{lpkh}$and K_{lpkh} defined by any one of the conditions (2.11) or (2.12).*

Theorem 2.8. *The associate curvature tensorsR_{lpkh} in$G\,R^v$- $R\,F_n$ coincides with the associate curvature tensorsK_{lpkh} in$G\,K^v$- $R\,F_n$and they are in terms of the associate torsion tensor $H_{kp.h}$.*

Contracting the indices i and h in (2.1) and (2.2), using (1.15c), (1.15e)and in view of (1.1), we get

(2.13) $$R_{jk}|_l = \lambda_l\, R_{jk} + (n-1)\mu_l\, g_{jk}$$

and

(2.14) $$K_{jk}|_l = \lambda_l\, K_{jk} + (n-1)\mu_l\, g_{jk} \quad ,$$

respectively.

The conditions (2.13) and (2.14) show that R-Ricci tensor R_{jk} and K-Ricci tensor K_{jk} cannot vanish, since the vanishing of any one of them would imply the vanishing of the covariant vector field μ_l, i.e. $\mu_l = 0$, a contradiction.

Thus, we conclude

Theorem 2.9. *In* $G\ R^v$-$R\ F_n$ *and* $G\ K^v$-$R\ F_n$ *,R-Ricci tensor* R_{jk} *and* K*-Ricci tensor* K_{jk}*are non-vanishing.*

Contracting the indices i and h in (2.3) and (2.4), using (1.13b), (1.12c), (1.12e) and in view of (1.1), we get

(2.15) $$H_k|_l = \lambda_l\, H_k + R_{lk} + (n-1)\mu_l\, y_k$$

and

(2.16) $$H_k|_l = \lambda_l\, H_k + K_{lk} + (n-1)\mu_l\, y_k \quad ,$$

respectively.

The conditions (2.15) and (2.16) show that the curvature vectorH_k cannot vanish, since the vanishing of any one of them would imply the vanishing of the covariant vector field μ_l, i.e. $\mu_l = 0$, a contradiction.

Thus, we conclude

Theorem 2.10. *In* $G\ R^v$-$R\ F_n$ *and* $G\ K^v$-$R\ F_n$ *, the curvature vector* H_k *is non-vanishing.*

The conditions (2.15) and (2.16) can be written as

(2.17) $$R_{lk} = H_k|_l - \lambda_l\, H_k - (n-1)\mu_l\, y_k$$

and

(2.18) $$K_{lk} = H_k|_l - \lambda_l\, H_k - (n-1)\mu_l\, y_k \quad ,$$

respectively.

Thus, we conclude

Theorem 2.11. *In* $G\ R^v$- $R\ F_n$ *and* $G\ K^v$- $R\ F_n$ *, R-Ricci tensor* R_{jk} *and K-Ricci tensor* K_{jk}*defined by any one of the conditions (2.17) or (2.18).*

Theorem 2.12. *R-Ricci tensor R_{jk} in $G\,R^v$- $R\,F_n$ coincides with K-Ricci tensor K_{jk} in $G\,K^v$- $R\,F_n$ and they are in terms of the curvature vector H_k .*

Transvecting the conditions (2.13) and (2.14) by y^k , using (1.7a), (1.13a), (1.13b) and (1.2a), we get

(2.19) $$R_j|_l = \lambda_l\, R_j + R_{jl} + (n-1)\mu_l\, y_j$$

and

(2.20) $$K_j|_l = \lambda_l\, K_j + K_{jl} + (n-1)\mu_l\, y_j \quad ,$$

respectively.

The conditions (2.19) and (2.20) show that the curvature vectors R_j and K_j cannot vanish, since the vanishing of any one of them would imply the vanishing of the covariant vector field μ_l , i.e. $\mu_l = 0$, a contradiction.

Thus, we conclude

Theorem 2.13. *In $G\,R^v$-$R\,F_n$ and $G\,K^v$-$R\,F_n$, the curvature vectors R_j and K_j are non-vanishing.*

The conditions (2.19) and (2.20) can be written as

(2.21) $$R_{jl} = R_j|_l - \lambda_l\, R_j - (n-1)\mu_l\, y_j$$

and

(2.22) $$K_{jl} = K_j|_l - \lambda_l\, K_j - (n-1)\mu_l\, y_j \quad ,$$

respectively.

Thus, we conclude

Theorem 2.14. *In $G\,R^v$- $R\,F_n$ and $G\,K^v$- $R\,F_n$, R-Ricci tensor R_{jl} and K-Ricci tensor K_{jl} defined by the conditions (2.21) and (2.22), respectively.*

Remark 2.1. In view of (2.17) and (2.18), both R-Ricci tensor R_{jl} *in* $G\,R^v$- $R\,F_n$ and K-Ricci tensor K_{jl} in $G\,K^v$- $R\,F_n$ are defined in terms of the curvature vector H_k (in sense of Berwald), different in the senses.

Remark 2.2. In view of (2.21), R-Ricci tensor R_{jl} is defined in terms of the curvature vector R_j (both Cartan's third curvature tensors R^i_{jkh}) and in view of (2.22), K-Ricci tensor K_{jl} is defined in terms of the curvature vector K_j (both Cartan's fourth curvature tensor K^i_{jkh}), similar in the senses.

Differentiating the conditions (2.3) and (2.4) partially with respect to y^j, using (1.11b) and (1.2c), we get

(2.23)

$$\dot{\partial}_j(H^i_{kh}|_l) = (\dot{\partial}_j\,\lambda_l)\,H^i_{kh} + \lambda_l(\,H^i_{jkh}) + \dot{\partial}_j R^i_{lkh} + (\dot{\partial}_j\mu_l)(\,\delta^i_h y_k - \delta^i_k y_h\,) + \mu_l(\,\delta^i_h g_{jk} - \delta^i_k g_{jh}\,)$$

and

(2.24)

$$\dot{\partial}_j(\,H^i_{kh}|_l) = (\dot{\partial}_j\,\lambda_l)\,H^i_{kh} + \lambda_l(\,H^i_{jkh}) + \dot{\partial}_j K^i_{lkh} + (\dot{\partial}_j\mu_l)(\,\delta^i_h y_k - \delta^i_k y_h\,) + \mu_l(\,\delta^i_h g_{jk} - \delta^i_k g_{jh}\,)\quad ,$$

respectively.

Using the commutation formula exhibited by (1.6) for the h(v) torsion tensor H^i_{jk} in the conditions (2.23) and (2.24) and using (1.10a), we get

(2.25)

$$H^i_{jkh}|_l + H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj}\,H^i_{rkh} = (\dot{\partial}_j\,\lambda_l)\,H^i_{kh}$$

$$+\,\lambda_l\,H^i_{jkh} + \dot{\partial}_j R^i_{lkh} + (\dot{\partial}_j\mu_l)(\,\delta^i_h y_k - \delta^i_k y_h\,) + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

and

(2.26)

$$H^i_{jkh}|_l + H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj}\,H^i_{rkh} = (\dot{\partial}_j\,\lambda_l)\,H^i_{kh}$$

$$+\,\lambda_l\,H^i_{jkh} + \dot{\partial}_j K^i_{lkh} + (\dot{\partial}_j\mu_l)(\,\delta^i_h y_k - \delta^i_k y_h\,) + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

.

The equations (2.25) and (2.26) together implies to

(2.27) $H^i_{jkh}|_l = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$

if and only if

(2.28) $H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj} H^i_{rkh}$
$= (\dot{\partial}_j \lambda_l) H^i_{kh} + \dot{\partial}_j R^i_{lkh} + (\dot{\partial}_j \mu_l)(\delta^i_h y_k - \delta^i_k y_h)$

and

(2.29) $H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj} H^i_{rkh}$
$= (\dot{\partial}_j \lambda_l) H^i_{kh} + \dot{\partial}_j K^i_{lkh} + (\dot{\partial}_j \mu_l)(\delta^i_h y_k - \delta^i_k y_h)$.

Thus, we conclude

Theorem 2.15.*In* $G\,R^v$*-R* F_n *, Berwald curvature tensor* H^i_{jkh} *is* $G\ v$*-R if and only if the condition (2.28) holds good.*

Theorem 2.16.*In* $G\,K^v$*-R* F_n *, Berwald curvature tensor* H^i_{jkh} *is* $G\ v$*-Rif and only if the condition (2.29) holds good.*

The condition (2.25) and (2.26) can be written as

$\dot{\partial}_j R^i_{lkh} = H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj} H^i_{rkh}$
$- (\dot{\partial}_j \lambda_l) H^i_{kh} - (\dot{\partial}_j \mu_l)(\delta^i_h y_k - \delta^i_k y_h)$

and

$\dot{\partial}_j K^i_{lkh} = H^r_{kh}(\dot{\partial}_j C^i_{lr}) - H^i_{rh}(\dot{\partial}_j C^r_{lk}) - H^i_{kr}(\dot{\partial}_j C^r_{lh}) + C^r_{lj} H^i_{rkh}$
$- (\dot{\partial}_j \lambda_l) H^i_{kh} - (\dot{\partial}_j \mu_l)(\delta^i_h y_k - \delta^i_k y_h)$,

respectively.

Thus, we conclude

Theorem 2.17. *The tensor*$(\dot{\partial}_j R^i_{lkh})$ *in*GR^v*-R*F_n *coincides withtensor*$(\dot{\partial}_j K^i_{lkh})$*in* GK^v*-R*F_n *provided the condition (2.27) holds.*

Let us consider a Finslerspace which Cartan's third curvature tensors$R^i_{jkh}$is satisfying the condition

(2.30) $R^i_{jkh} = K(\delta^i_h g_{jk} - \delta^i_k g_{jh})$.

The space characterized by the condition (2.30) is called *h-isotropic* [7]. It is to be noted that the constant for the concept of h-isotropic does not coincide with that constant curvature due to Berwald. For an h-isotropic space, K is constant. Therefore for the space considered K is constant.

Taking the v-covariant derivative for the condition (2.30) with respect to y^l, using (1.7b), we get

$R^i_{jkh}|_l = K\left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right)$.

In view of the condition (2.1), the above equation becomes

$\lambda_l R^i_{jkh} = (K - \mu_l)\left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right)$

which can be written as

(2.31) $R^i_{jkh} = \beta\left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right)$,

where $\beta = \frac{(K - \mu_l)}{\lambda_l}$.

Theorem 2.18.*In$G\ R^v$-$R\ F_n$,the h-isotropic space is characterized by the condition (2.31).*

3.Decomposition of the Curvature Tensors $\overset{+}{R}{}^i_{jkh}(x,y)$ and $\overset{+}{K}{}^i_{jkh}(x,y)$ in a Finsler Space Equipped with Non-Symmetric Connection

We shall discuss some of the decompositions for the curvature tensors field $\overset{+}{R}{}^i_{jkh}(x,y)$ and $\overset{+}{K}{}^i_{jkh}(x,y)$ in a Finsler space equipped with non-symmetric connection for Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} .

G. H. Vranceanu [21] has defined a non-symmetric connection $\left(\Gamma^{*i}_{jk} \neq \Gamma^{*i}_{kj}\right)$ in n-dimensional Finsler space F_n . Let consider an n-dimensional Finsler space F_n with non-symmetric connection $\left(\Gamma^{*i}_{jk} \neq \Gamma^{*i}_{kj}\right)$ which is based on a non-symmetric fundamental tensor $g_{ij}(x,y) \neq g_{ji}(x,y)$. Let write

(3.1) $\Gamma^{*i}_{jk} = \mathrm{M}^{*i}_{jk} + \frac{1}{2}\mathrm{N}^{*i}_{jk}$, where M_{jk}^{*i} and $\frac{1}{2}\mathrm{N}_{jk}^{*i}$ are respectively the symmetric and skew-symmetric parts of Γ_{jk}^{*i} .

We introduce another connection coefficient $\Gamma_{kj}^{*i}(x,y)$ defined as order

(3.2) $$\bar{\Gamma}_{jk}^{*i} = \mathrm{M}_{jk}^{*i} - \frac{1}{2}\mathrm{N}_{jk}^{*i} \quad .$$

With the help of the conditions (4.1) and (4.2), we get

$$\Gamma_{jk}^{*i}(x,y) = \bar{\Gamma}_{jk}^{*i}(x,y) \quad .$$

Following É. Cartan [2], let a vertical stroke $|_j$, follow by an index denote covariant derivative with respect to y, the covariant derivative of any contravariant vector field $X^i(x,y)$ with respect to y^j is defined as follows:

(3.3) $$X^{i+}|_j := \dot{\partial}_j X^i + X^r C_{rj}^i \quad ,$$

where a positive sign below an index and following by a vertical stroke indicates that the covariant derivative has been formed with respect to the connection Γ_{kj}^{*i} as for as that index is concerned. The covariant derivative defined in (3.3) is called ⊕-*covariant differentiation* of $X^i(x,y)$ with respect to y^j also is called *v*−*covariant differentiation* (*Cartan's first kind covariant differentiation*).

The entities $\overset{+}{R}{}_{jkh}^i$ and $\overset{+}{K}{}_{jkh}^i$ are called *the curvature tensors* (with respect to the⊕-covariant derivative) of Finsler space with respect to the non-symmetric connection Γ_{jk}^{*i} such that

$$\overset{+}{R}{}_{jkh}^i = \dot{\partial}_h \Gamma_{jk}^{*i} + \left(\dot{\partial}_s \Gamma_{jk}^{*i}\right) G_h^s + C_{jm}^i \left(\dot{\partial}_k G_h^m - G_{kl}^m G_h^l\right) + \Gamma_{mk}^{*i}\, \Gamma_{jh}^{*m} - \dot{\partial}_k \Gamma_{jh}^{*i} - \left(\dot{\partial}_s \Gamma_{jh}^{*i}\right) G_k^s - C_{jm}^i \left(\dot{\partial}_h G_k^m - G_{hl}^m G_k^l\right) - \Gamma_{mh}^{*i}\, \Gamma_{jk}^{*m}$$

and

$$\overset{+}{K}{}_{jkh}^i := \partial_h \Gamma_{jk}^{*i} + \left(\dot{\partial}_s \Gamma_{jh}^{*i}\right) G_k^s + \Gamma_{mh}^{*i} \Gamma_{kj}^{*m} - \partial_h \Gamma_{jk}^{*i} - \left(\dot{\partial}_s \Gamma_{jh}^{*i}\right) G_k^s - \Gamma_{mh}^{*i} \Gamma_{kj}^{*m}$$

These curvature tensors $\overset{+}{R}^i_{jkh}$ and $\overset{+}{K}^i_{jkh}$ are satisfying the following:

(3.4) a) $\overset{+}{R}^i_{jkh}y^j = \overset{+}{R}^i_{kh}$, b) $\overset{+}{R}^i_{jki} = \overset{+}{R}_{jk}$,

c) $\overset{+}{K}^i_{jkh}y^j = \overset{+}{K}^i_{kh}$ and d) $\overset{+}{K}^i_{jki} = \overset{+}{K}_{jk}$.

Henceforth a Finsler space $F_n$equipped with non-symmetric connection will be written as F_n^*.

A Finsler space F_n^* is said to be *a generalized*$\overset{+}{R}^v$- *non symmetric recurrent space* and *a generalized*$\overset{+}{K}^v$- *non symmetric recurrent space* when Cartan's curvature tensors field $\overset{+}{R}^i_{jkh}(x,y)$ and $\overset{+}{K}^i_{jkh}(x,y)$ are satisfingthe following conditions

(3.5) $$\overset{+}{R}^i_{jkh}|_l = \lambda_l \overset{+}{R}^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}) ,$$

$\overset{+}{R}^i_{jkh} \neq 0$

and

(3.6) $$\overset{+}{K}^i_{jkh}|_l = \lambda_l \overset{+}{K}^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}) ,$$

$\overset{+}{K}^i_{jkh} \neq 0$,

respectively. We shall denote them briefly by $G\,\overset{+}{R}^v$-RF_n^* and $G\,\overset{+}{K}^v$-RF_n^* , respectively.

Transvecting the conditions (3.5) and (3.6) by y^j , using (1.7a), (3.4a), (1.2a), (3.4c) and in view of (1.1), we get

(3.7) $$\overset{+}{R}^i_{kh}|_l = \lambda_l \overset{+}{R}^i_{kh} + \overset{+}{R}^i_{lkh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h)$$

and

(3.8) $$\overset{+}{K}^i_{kh}|_l = \lambda_l \overset{+}{K}^i_{kh} + \overset{+}{K}^i_{lkh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h) ,$$

respectively.

Thus, we conclude

Theorem 3.1. *In* $G\overset{+}{R}^{v}$-RF_n^* *and* $G\overset{+}{K}^{v}$-RF_n^**, the v-covariant derivative of the first order for the torsion tensors* $\overset{+}{R}^i_{kh}$ *and* $\overset{+}{K}^i_{kh}$ *given by the conditions (3.7) and (3.8), respectively.*

The conditions (3.7) and (3.8) can be written as

(3.9) $$\overset{+}{R}^i_{lkh} = \overset{+}{R}^i_{kh}|_l - \lambda_l \overset{+}{R}^i_{kh} - \mu_l(\delta^i_h y_k - \delta^i_k y_h)$$

and

(3.10) $$\overset{+}{K}^i_{lkh} = \overset{+}{K}^i_{kh}|_l - \lambda_l \overset{+}{K}^i_{kh} - \mu_l(\delta^i_h y_k - \delta^i_k y_h) ,$$

respectively.

Thus, we conclude

Theorem 3.2. *In* $G\overset{+}{R}^{v}$- $R\,F_n^*$ *and* $G\overset{+}{K}^{v}$- $R\,F_n^*$ *, the curvature tensors* $\overset{+}{R}^i_{jkh}$ *and* $\overset{+}{K}^i_{jkh}$ *defined by the conditions (3.9) and (3.10), respectively.*

Contracting the indices i and h in the conditions (3.5) and (3.6), using (3.4b), (3.4d) and in view of (1.1), we get

(3.11) $$\overset{+}{R}_{jk}|_l = \lambda_l \overset{+}{R}_{jk} + (n-1)\mu_l\, g_{jk}$$

and

(3.12) $$\overset{+}{K}_{jk}|_l = \lambda_l \overset{+}{K}_{jk} + (n-1)\mu_l\, g_{jk} ,$$

respectively.

The conditions (3.11) and (3.12) show that $\overset{+}{R}$-Ricci tensor $\overset{+}{R}_{jk}$ and $\overset{+}{K}$-Ricci tensor $\overset{+}{K}_{jk}$ cannot vanish, since the vanishing of any one of them would imply the vanishing of the covariant vector field μ_l , i.e. $\mu_l = 0$, a contradiction.

Thus, we conclude

Theorem 3.3. *In* $G\overset{+}{R}^{v}$- $R\,F_n^*$ *and* $G\overset{+}{K}^{v}$- $R\,F_n^*$ *, the v-covariant derivative of the first order for* $\overset{+}{R}$*-Ricci tensor* $\overset{+}{R}_{jk}$ *and* $\overset{+}{K}$*-Ricci tensor* $\overset{+}{K}_{jk}$ *are non-vanishing.*

Now, let usconsider the decomposability of the curvature tensor field $\overset{+}{R}^i_{jkh}$ in a Finsler space F_n^* , since the curvature tensor under consideration is a mixed tensor of rank 4, hence it may be written either as a tensor product of a vector and a tensor of rank 3 or as a tensor product of two tensors each of rank 2. In the first case, the possibilities forms of decomposition for the curvature tensor $\overset{+}{R}^i_{jkh}$ are as follows:

(3.13) a) $\overset{+}{R}^i_{jkh} = X^i \, \Psi_{jkh}$, b) $\overset{+}{R}^i_{jkh} = X_j \, \Psi^i_{kh}$,

c) $\overset{+}{R}^i_{jkh} = X_k \, \Psi^i_{jh}$ and d) $\overset{+}{R}^i_{jkh} = X_h \, \Psi^i_{jk}$.

In the second case the possibilities are as follows:

(3.14) a) $\overset{+}{R}^i_{jkh} = q^i_j \, \Psi_{kh}$, b) $\overset{+}{R}^i_{jkh} = q^i_k \, \Psi_{jh}$ and c) $\overset{+}{R}^i_{jkh} = q^i_h \Psi_{jk}$.

Similarly, the possibilities form of decomposition for the curvature tensor $\overset{+}{K}^i_{jkh}$ are as follows:

(3.15) a) $\overset{+}{K}^i_{jkh} = X^i \, \Psi_{jkh}$, b) $\overset{+}{K}^i_{jkh} = X_j \, \Psi^i_{kh}$,

c) $\overset{+}{K}^i_{jkh} = X_k \, \Psi^i_{jh}$ and d) $\overset{+}{K}^i_{jkh} = X_h \, \Psi^i_{jk}$.

In the second case the possibilities are as follows:

(3.16) a) $\overset{+}{K}^i_{jkh} = q^i_j \, \Psi_{kh}$, b) $\overset{+}{K}^i_{jkh} = q^i_k \, \Psi_{jh}$ and c) $\overset{+}{K}^i_{jkh} = q^i_h \Psi_{jk}$.

Out of several possibilities given by (3.13), (3.14), (3.15) and (3.16), our goal is to study the possibilities given by (3.13a), (3.13b), (3.15a) and (3.15b).

Suppose that Cartan's third curvature tensor $\overset{+}{R}{}^i_{jkh}$ and Cartan's fourth curvature tensor $\overset{+}{K}{}^i_{jkh}$ are decomposed in the forms (3.13a) and (3.15a), respectively.

Taking the v-covariant derivative of the forms (3.13a) and (3.15a) with respect to y^l, we get

(3.17) $$\overset{+}{R}{}^i_{jkh}|_l = X^i|_l \, \Psi_{jkh} + X^i \overset{+}{\Psi}_{jkh}|_l$$

and

(3.18) $$\overset{+}{K}{}^i_{jkh}|_l = X^i|_l \, \Psi_{jkh} + X^i \overset{+}{\Psi}_{jkh}|_l \quad ,$$

respectively.

Using the conditions (3.5) and (3.6) in (3.17) and (3.18), respectively, we get

$$\lambda_l \overset{+}{R}{}^i_{jkh} + \mu_l (\delta^i_h g_{jk} - \delta^i_k g_{jh}) = X^i|_l \, \Psi_{jkh} + X^i \overset{+}{\Psi}_{jkh}|_l$$

and

$$\lambda_l \overset{+}{K}{}^i_{jkh} + \mu_l (\delta^i_h g_{jk} - \delta^i_k g_{jh}) = X^i|_l \, \Psi_{jkh} + X^i \overset{+}{\Psi}_{jkh}|_l \quad .$$

In view of (3.13a) and (3.15a) and if the decomposable vector X^isupposed to be a covariant constant, then from (3.17) and (3.18) together, we immediately get

$$\lambda_l X^i \, \Psi_{jkh} + \mu_l (\delta^i_h g_{jk} - \delta^i_k g_{jh}) = X^i \overset{+}{\Psi}_{jkh}|_l$$

which can be written as

(3.19) $$\overset{+}{\Psi}_{jkh}|_l = \lambda_l \, \Psi_{jkh} + \phi_l (\delta^i_h g_{jk} - \delta^i_k g_{jh}) \quad ,$$ where $\phi_l = \frac{\mu_l}{X^i}$.

Thus, we conclude

Theorem 3.4.*In* $G \overset{+}{R}{}^v$*-*RF_n^* *and* $G \overset{+}{K}{}^v$*-*RF_n^**, the decomposable tensor field* $\Psi_{jkh}(x, y)$ *is generalized recurrent if the decomposable vector* X^i *assumed to be a covariant constant.*

Transvecting (3.19) by y^j, using (1.7a) and in view of (1.1), we get

(3.20) $$\overset{+}{\Psi}_{kh}|_l = \lambda_l\, \Psi_{kh} + \Psi_{lkh} + \phi_l\left(\delta^i_h y_k - \delta^i_k y_h\right) \quad ,$$

where $\Psi_{jkh}\, y^j = \Psi_{kh}$ and $\dot{\partial}_j\, \Psi_{kh} = \Psi_{jkh}$.

Thus, we conclude

Theorem 3.5.*In* $G\overset{+}{R}{}^v$*-*RF_n^* *and* $G\overset{+}{K}{}^v$*-*RF_n^**, the v-covariant derivative for the decomposable tensor field* $\Psi_{kh}(x,y)$ *is given by the condition (3.20), if the decomposable vector* X^i *assumed to be a covariant constant.*

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حول تعميم فضاءات غير متماثلة أحادية المعاودة

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الملخص

في هذه الورقة ، عرفنا فضاء فنسلر F_n الذي يكون فيه تقوس كارتان الثالث R^i_{jkh} والرابع K^i_{jkh} يحققان في مفهوم كارتان العلاقتين الآتيتين :

$$R^i_{jkh}|_l = \lambda_l R^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}) \quad , \quad R^i_{jkh} \neq 0 \quad ,$$

$$K^i_{jkh}|_l = \lambda_l K^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}) \quad , \quad K^i_{jkh} \neq 0 \quad ,$$

حيث $|_l$ هي مشتقة كارتان من النوع الأول بالنسبة إلى المسقط الوضعي y^l، λ_l وμ_l هي حقول غير صفرية لمتجهات متحدة الاختلاف وأطلقنا على هذه الفضاءات تعميم فضاء فنسلر R^v -أحادي المعاودة و تعميم فضاء فنسلر K^v -أحادي المعاودة ورمزنا لهم بالرموز التالية $R\ F_n GR^v$ - و $R\ F_n GK^v$ -على التوالي، وكذلك تم إيجاد العديد من الصيغ ، المبرهنات والمتطابقات المختلفة لهذه التقوسات في هذا الفضاءات، وأثبتنا بان الكثير من هذه التقوسات في هذه الفضاءات لا تنتهي، كذلك قدمنا تعريف التعميمات للموترات التقوسية في فضاء فنسلر، التي يكون فيها كل من الموتر التقوسي الثالث $\overset{+}{R}{}^i_{jkh}$ والرابع $\overset{+}{K}{}^i_{jkh}$ لكارتان بالنسبة لرابطه Γ^{*i}_{jk} غير المتماثلة أحادية المعاودة المعممة وتم الحصول على العديد من المتطابقات والمبرهنات المختلفة ذات الصلة بهذا الفضاء، كذلك أثبتنا بأن تقوسات رتشي $\overset{+}{R}_{kh}$ و$\overset{+}{K}_{kh}$ لكل من الموتر التقوسي الثالث $\overset{+}{R}{}^i_{jkh}$ والرابع $\overset{+}{K}{}^i_{jkh}$ لكارتان لا تنتهي في هذا الفضاء.

كلمات مفتاحية: تعميم فضاء فنسلر R^v - أحادي المعاودة، تعميم فضاء فنسلر K^v - أحادي المعاودة، تعميم فضاء فنسلر $\overset{+}{R}{}^v$ - أحادي المعاودة غير المتماثل، تعميم فضاء فنسلر $\overset{+}{K}{}^v$ - أحادي المعاودة غير المتماثل.